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Shapley Value Approximation for Games with Distant Players

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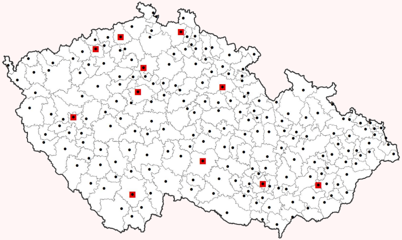
Motivation

Waste Management Game

Game-theoretic model of a potential waste management situation in the Czech Republic

Study of cooperation among 206 waste producers minimizing total cost for waste disposal



Osička, O.: Game Theory in Waste Management. Master's thesis, Brno University of Technology (2016)

Cooperative Game

(N, v) a cooperative game

$N = \{p_1, \ldots, p_n\}$ set of n players

v characteristic function

$v(S)$ total cost, coalition S is able to achieve

$\varphi(N, v) = (\varphi_{p_1}(N, v), \ldots, \varphi_{p_n}(N, v))$ the Shapley value

The Shapley Value

$$\varphi(N, v) = (\varphi_{p_1}(N, v), \ldots, \varphi_{p_n}(N, v))$$

$$\varphi_{p_i}(N, v) = \sum_{S \subseteq N:\, p_i \in S} \frac{(|S| - 1)!\,(|N| - |S|)!}{|N|!} (v(S) - v(S \setminus \{p_i\}))$$

The characteristic function values for approximately $1.03 \cdot 10^{62}$ coalitions needed for the waste management game in the Czech Republic

Shapley Value Approximation

d_{crit} critical distance

beyond this distance, the cooperation expected to have no impact

c_{max} maximum number of cooperating players

Algorithm

Algorithm

Step 1
: Determination of all reasonable distances

Step 2
: Determination of the set of all reasonable coalitions

Step 3
: Modified Shapley value computation

Step 4
: Final refinements

Step 1

$d_{i,j}$ distance between players p_i and p_j

Compared with the critical distance d_{crit}

$$a_{i,j} = \begin{cases} 1 & \text{if} \quad d_{i,j} \leq d_{crit} \\ 0 & \text{otherwise} \end{cases} .$$

Step 2

Question, if $\{p_i, p_j, p_k\}$ should be included when $a_{i,j} = 1$, $a_{j,k} = 1$, but $a_{i,k} = 0$

```
if c_max ≥ 1 then
    for j = 1 to c_max do
        set C_j = ∅
    end for
    for i = 1 to number of players do
        add {p_i} to C_1
    end for
    for j = 2 to c_max do
        for all S ∈ C_{j−1} do
            for i = 1 to number of players do
                if p_i ∉ S and Σ_{k: p_k ∈ S} a_{i,k} ≥ 1 then
                    add S ∪ {p_i} to C_j
                end if
            end for
        end for
    end for
    set C = ⋃_{j=1}^{c_max} C_j
end if
add ∅ and N to C
```

Step 3

$$\psi'_{p_i}(N, v, C) = \sum_{S \in C:\, p_i \in S} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!} \left(v(S) - v'(S \setminus \{p_i\})\right)$$

$$v'(S \setminus \{p_i\}) = \begin{cases} v(S \setminus \{p_i\}) & \text{if} \quad S \setminus \{p_i\} \in C \\ v_{min}(S \setminus \{p_i\}) & \text{otherwise} \end{cases}$$

Step 3

$$
\begin{aligned}
v_{min}(S \setminus \{p_i\}) = \min_{x_j :\, j \in J} & \sum_{j \in J} v(T_j) x_j, \\
\text{s. t.} \quad & \bigcup_{j \in J :\, x_j = 1} T_j = S \setminus \{p_i\}, \\
& \bigcap_{j \in J :\, x_j = 1} T_j = \varnothing, \\
& x_j \in \{0, 1\} \qquad \forall j \in J,
\end{aligned}
$$

where $C = \{T_1, \ldots, T_{|C|}\}$ and $J = \{1, \ldots, |C|\}$

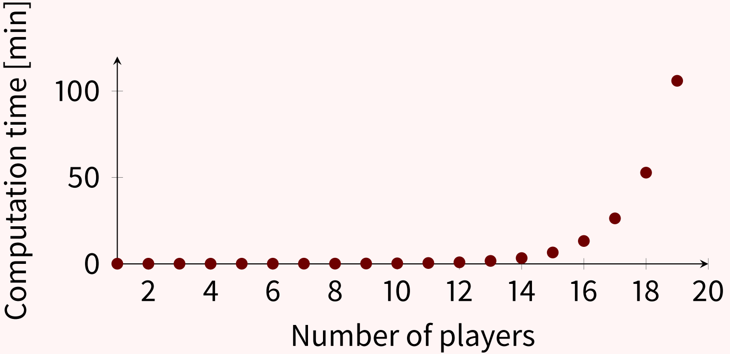
Step 4

Total cost completely divided among the players

$$\psi_{p_i}(N, v, C) = \psi'_{p_i}(N, v, C) + \frac{v(N) - \sum_{p_i \in N} \psi'_{p_i}(N, v, C)}{n}.$$

Computation Time

Classical Approach



For 206 players approximately $3.95 \cdot 10^{52}$ years

Including the questioned coalition

		c_{max}		
		5	6	7
d_{crit}	0	1 min 15 s	1 min 12 s	1 min 15 s
	10	1 min 17 s	1 min 18 s	1 min 18 s
	20	3 min 52 s	4 min 13 s	4 min 35 s
	30	38 min 3 s	1 h 30 min 39 s	4 h 23 min 9 s
	40	5 h 3 min 7 s	16 h 10 min 32 s	59 h 5 min 13 s
	50	24 h 4 min 47 s	–	–

Not incl. the questioned coalition

		c_{max}		
		5	10	15
d_{crit}	0	1 min 9 s	1 min 7 s	1 min 9 s
	10	1 min 11 s	1 min 11 s	1 min 11 s
	20	1 min 56 s	1 min 57 s	1 min 56 s
	30	4 min 23 s	4 min 22 s	4 min 23 s
	40	11 min 47 s	11 min 55 s	11 min 48 s
	50	38 min 17 s	45 min 47 s	45 min 11 s
	60	2 h 2 min 15 s	3 h 33 min 16 s	3 h 33 min 32 s
	70	5 h 54 min 33 s	19 h 20 min 55 s	19 h 42 min 2 s

Accuracy

Algorithm	2.1	2.1	2.1	2.2	2.2
c_{max}	1	5	7	5	5
d_{crit}	0	20	20	40	70
Player 1	44,810,577	44,807,134	44,806,050	44,807,393	44,806,007
Player 2	283,698	283,929	284,036	283,971	283,933
Player 3	3,392,713	3,392,944	3,393,051	3,392,986	3,393,097
Player 4	772,546	772,776	772,884	772,818	772,859
Player 5	488,915	489,146	489,253	489,188	489,257
Player 6	1,056,360	1,056,591	1,056,698	1,056,632	1,056,641
Player 7	252,419	252,650	252,757	252,691	252,744
Player 8	309,399	309,630	309,737	309,671	309,754
Player 9	938,014	938,245	938,352	938,286	938,395
Player 10	602,500	602,731	602,838	602,773	602,883

Conclusion

Closing Thoughts

Reasonable time

Only for games in which the critical distance can be determined

More extensions to recognize the reasonable coalition could be performed