Cooperation of customers in traveling salesman problems with profits*

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Abstract

In the profitable tour problem, the carrier decides whether to visit a particular customer with respect to the prize the customer offers for being visited and traveling cost associated with the visit, all in the context of other customers. Our focus is on the prizes customers need to offer to ensure being visited by the carrier. This can be formulated as a cooperative game where customers may form coalitions and make decisions on the prize values cooperatively. We define the profitable tour game describing this situation and analyze the cost associated with each coalition of customers and prizes that help to achieve it. We derive properties of the optimal prizes to be offered when the grand coalition is formed. These properties describe relationship between the prizes and the underlying traveling salesman game to provide connection with extensive literature on core allocations in traveling salesman games. The most important result states that the set of optimal prizes coincides with the core of the underlying traveling salesman game if this core is nonempty.

Keywords: Traveling salesman problem; Profitable tour problem; Prize-collecting TSP; Logistics; Cooperative game theory; Prize allocation

1 Introduction

The traveling salesman problem (TSP) is one of the most studied problems in logistics [9]. It answers the question of how to visit several places within a single tour starting and finishing at a particular place while minimizing the total traveling cost. Throughout this paper, we will use words *customers* to refer to the places to be visited and *carrier* to refer to their visitor. There have been a lot of variants of the traveling salesman problem dealing with different aspects of the underlying situation. For example, the customers might offer prizes for being visited. This changes the carrier's goal from pure minimization of the traveling cost

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to minimization of the traveling cost and maximization of the profit based on the collected prizes at the same time. There might be additional requirements such that, for example, a certain number of customers must be visited. In general, such problems are called the traveling salesman problems with profits [4]. According to Feillet et al. [4], the variant without additional requirements, that is maximizing the difference between the prize-based profit and the traveling cost, is called the profitable tour problem (PTP).

A cooperative formulation of the traveling salesman problem, the traveling salesman game, attracted scientific interest after a question proposed by Fishburn and Pollak [5]. After a road trip of a visitor visiting several sponsors, how should they be charged in a fair manner such that they cover the total cost of the trip? To find such allocation, Fishburn and Pollak [5] stated several conditions on the allocated costs, then Potters et al. [10] provided game-theoretic insights. Nowadays, there exist many studies focusing on such allocations [2, 8, 14, 13, 15]. In fact, traveling salesman games are the main topic of a great share of the atricles gathered by a recent survey on cost allocation methods used in collaborative transportation [7].

Estévez-Fernández et al. [3] proposed a traveling salesman game alternative for the case of customers offering prizes, called the routing game with revenues. Its focus remains on the allocation of the total cost of a tour visiting all customers. However, the total cost incurred by the customers is in fact the sum of all offered prizes. As shown by the following example, sometimes it is needed to allocate more than the total traveling cost of the tour.

Example 1. Let $1, \dots, 6$ be customers which desire to be visited by a carrier departing from and returning to depot 0 as illustrated in Figure 1, an example taken from [15]. All edges of

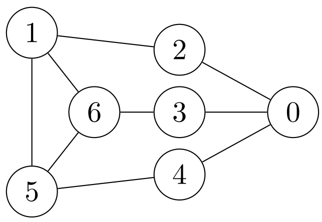

Figure 1: Problem in Example 1

the graph represent unit traveling cost such that the minimal cost of traveling from 0 to 4 is of one unit (0-4) whereas the minimal cost of traveling from 2 to 3 is of two units (2-0-3). It is easy to see that the least costly tour would be of 8 units (0-4-5-6-3-6-1-2-0). Looking at the problem from a perspective of a prize-collecting carrier, let us assume the carrier is originally visiting all of the customers which implies the aforementioned cost of 8 units. If the carrier decides to drop customers 4 and 5 and only performs tour 0-3-6-1-2-0, the associated cost drops to 5 units. This means that customers 4 and 5 can make themselves worth visiting by offering a total prize of at least 3 units to cover the additional traveling cost associated with their visit. The same requirement of at least 3 units being offered could be derived for the pair of customers 3 and 6 and the pair of customers 1 and 2. Together the prizes of all customers must add up to at least 9 units. Otherwise, some customers do not get visited. In other words, at a cost of 8 units, the coalition of all customers fails to fulfill its purpose and both the traveling salesman game and the routing game with revenues do not describe such a situation properly as they assume all customers being visited at this cost.

The purpose of this paper is to define the profitable tour game, a cooperative version of the profitable tour problem, and to derive its properties. We are particularly interested in prize allocations that create incentives for the carrier to visit all relevant customers.

The traveling salesman game could be interpreted as a problem where the carrier is forced to visit all customers and suggest charges in a way that makes the customers willing to be visited. On the other hand, the profitable tour game allows the carrier to have a final word and introduces a problem where the customers need to compete or cooperate and make themselves worth being visited. The profitable tour game is not only of theoretical interest as it relates to many situations occurring in practice. In [2], for example, the Logistics Department at Norsk Hydro Olje AB determines how gas stations across southern Sweden should be charged for distribution of gas utilizing the concept of the traveling salesman game. If another gas station would like to join the initial set of stations, a simple question of what charges should the gas station expect for being even considered interesting already requires a point of view as given by the profitable tour game. Other related applications include, for example, customer selection in less-than-truckload transportation [1]. When it comes to delivery and pickup of goods, the customers might need to induce the carrier to visit them by offering sufficient rewards. Subsequently, negotiation with other customers in the same position could lead to better prizes while the carrier's visit would remain guaranteed.

The remainder of this paper is organized as follows. In section 2, we analyze the costs associated with different coalitions of customers and define the profitable tour game. Section 3 studies the optimal prizes to be offered by customers. Concluding remarks follow in section 4.

2 From the traveling salesman game to the profitable tour game

The problem outlined in the introduction is a two-stage conflict of $n+1$ decision-makers. First, n customers decide on prizes offered to a carrier for being visited and, after all prizes are observed, the carrier decides which customers to visit and how to perform the tour. We assume the carrier to be rational and profit-maximizing, which means, the strategy is to choose the tour with the largest difference between the prize-based profit and the traveling cost. With this in mind, the customers want to set the prizes in a least costly manner that still guarantees them to be visited. It might be useful to form coalitions with other customers. Such coalitions then aim to set prizes offered by their customers such that their sum is the least possible to guarantee being visited regardless of the other customers' offered prizes.

The whole situation is a zero-sum game, that is, whatever is paid out by the customers gets collected by the carrier. This does not offer opportunities for a meaningful cooperation of all players. On the contrary, leaving out the carrier and focusing on the conflict of customers only, the prizes can be set in way that benefits other customers as well. This is the idea for defining the profitable tour game as a cooperative game of the customers.

For modeling purposes, we impose standard assumptions on the traveling costs c_{ij} among customers themselves and between them and the carrier's home depot. Denoting the set of all customers by N and the home depot by 0, these assumptions are

$$c_{ii} = 0 \qquad \forall i \in N \cup \{0\}, \tag{1}$$

$$c_{ij} \leq c_{ik} + c_{kj} \qquad \forall i, j, k \in N \cup \{0\}. \tag{2}$$

Assumptions (1) imply no traveling cost is associated with staying in the same place and the triangle inequalities (2) make sure that the costs always represent the lowest possible costs which cannot be beaten by going another way. These assumptions are common in the literature. There are studies of traveling salesman problems with symmetric as well as asymmetric traveling costs. We don't limit our focus by imposing assumptions on this symmetry.

2.1 The traveling salesman game

To define the profitable tour game and derive its properties, it comes in handy to utilize the definition of the traveling salesman game by Potters et al. [10] which will become our starting point.

Let $N = \{1, \ldots, n\}$ be the set of all customers. For each group of customers $S \subseteq N$ which need to be visited from depot 0 using only one vehicle starting and finishing in the depot, the least total traveling cost could be obtained by solving the *traveling salesman problem* (TSP) given by

$$\mathrm{Cost}^{TSP}(S) = \min_{x_{ij}} \sum_{i \in S \cup \{0\}} \sum_{j \in S \cup \{0\}} c_{ij} x_{ij} \tag{3}$$

$$\text{s.t.} \quad \sum_{i \in S \cup \{0\}} x_{ij} = 1 \qquad \forall j \in S \cup \{0\}, \tag{4}$$

$$\sum_{j \in S \cup \{0\}} x_{ij} = 1 \qquad \forall i \in S \cup \{0\}, \tag{5}$$

$$\sum_{i \in T} \sum_{j \in T} x_{ij} \leq |T| - 1 \qquad \forall T \subset S \cup \{0\} \colon T \neq \varnothing, \tag{6}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in S \cup \{0\}. \tag{7}$$

In this integer linear programming model, x_{ij} is a binary variable that indicates whether the vehicle travels directly from i to j in the final tour. Constraints (4) and (5) ensure that all customers in S are visited and constraints (6) eliminate any subtours to guarantee the solution to be a single tour. The binary nature of x_{ij} is prescribed by (7). A tour with the lowest traveling cost is then selected by (3).

The pair (N, Cost^{TSP}) is then the traveling salesman game as defined by Potters et al. [10].

2.2 Cooperation in the profitable tour problem

The introduction of prizes offered by customers to the carrier for visiting them, denoted by Prize_j for the prize offered by customer $j \in N$, requires a slightly different view of the problem. Finding the optimal profit of the carrier is then the *profitable tour problem* (PTP) which can be formulated as

$$\text{Profit}^{PTP} = \max_{x_{ij},\delta_T} \left(\sum_{i\in N\cup\{0\}} \sum_{j\in N} \text{Prize}_j x_{ij} - \sum_{i\in N\cup\{0\}} \sum_{j\in N\cup\{0\}} c_{ij}x_{ij} \right) \tag{8}$$

$$\text{s.t.} \quad \sum_{i\in N\cup\{0\}} \sum_{j\in N\cup\{0\}} x_{ij} \leq M \sum_{j\in N} x_{0j}, \tag{9}$$

$$\sum_{i\in N\cup\{0\}} x_{ij} = \sum_{k\in N\cup\{0\}} x_{jk} \qquad \forall j \in N \cup \{0\}, \tag{10}$$

$$\sum_{i\in T} \sum_{j\in T} x_{ij} \leq |T| - 1 + \delta_T M \quad \forall T \subset N \cup \{0\} \colon T \neq \varnothing, \tag{11}$$

$$\sum_{i\in (N\cup\{0\})\setminus T} \sum_{j\in N\cup\{0\}} (x_{ij} + x_{ji}) \leq (1-\delta_T)M \qquad \forall T \subset N \cup \{0\} \colon T \neq \varnothing, \tag{12}$$

$$x_{ij} \in \{0,1\} \qquad \forall i,j \in N \cup \{0\}, \tag{13}$$

$$\delta_T \in \{0,1\} \qquad \forall T \subset N \cup \{0\} \colon T \neq \varnothing, \tag{14}$$

where x_{ij} again indicates whether the vehicle travels directly from i to j, δ_T is a binary variable activating the subtour elimination constraints, and M is a big enough number (for example, $M = |N \cup \{0\}|^2$ would be sufficient).

If there are any customers to be visited, constraint (9) ensures that the tour starts in the depot. Constraints (10) ensure that if the vehicle arrives at a certain customer it needs to continue its tour afterwards. For the depot, the intuition is that the tour needs to both start and finish there. Constraints (11) are the subtour elimination constraints, thanks to the combination of δ_T and M, activated only when necessary. This is then decided by constraints (12). Assuming a set of customers $S \subseteq N$ to be visited in the optimal solution, the activation works by ensuring δ_T to equal 0 for all nonempty $T \subset S \cup \{0\}$. Constraints (13) and (14) state the binary nature of variables x_{ij} and δ_T. Overall, a tour maximizing the difference between the prize-based profit and the traveling cost is chosen by (8).

In the case of PTP, the carrier is not forced to visit all customers, but visits only the most profitable subset of them. On the other hand, if all customers in coalition $S \subseteq N$ (and only those) needed to be visited, new constraints could be introduced in the PTP model which would create what we refer to as the *profitable tour problem with all customers visited* (PTP-AV). This can be formulated generally for any group of customers S as

$$\text{Profit}^{AV}(S) = \max_{x_{ij},\delta_T} \left(\sum_{i\in S\cup\{0\}} \sum_{j\in S} \text{Prize}_j x_{ij} - \sum_{i\in S\cup\{0\}} \sum_{j\in S\cup\{0\}} c_{ij}x_{ij} \right) \tag{15}$$

$$\text{s.t.} \quad \sum_{i\in S\cup\{0\}} x_{ij} = \sum_{k\in S\cup\{0\}} x_{jk} \qquad \forall j \in S\cup\{0\}, \tag{16}$$

$$\sum_{i\in S\cup\{0\}} x_{ij} = 1 \qquad \forall j \in S\cup\{0\}, \tag{17}$$

$$\sum_{i\in S\cup\{0\}} \sum_{j\in S\cup\{0\}} x_{ij} \leq M \sum_{j\in S} x_{0j}, \tag{18}$$

$$\sum_{i\in T} \sum_{j\in T} x_{ij} \leq |T| - 1 + \delta_T M \quad \forall T \subset S\cup\{0\}\colon T \neq \varnothing, \tag{19}$$

$$\sum_{i\in(S\cup\{0\})\setminus T} \sum_{j\in S\cup\{0\}} (x_{ij} + x_{ji}) \leq (1-\delta_T)M \qquad \forall T \subset S\cup\{0\}\colon T \neq \varnothing, \tag{20}$$

$$x_{ij} \in \{0,1\} \qquad \forall i,j \in S\cup\{0\}, \tag{21}$$

$$\delta_T \in \{0,1\} \qquad \forall T \subset S\cup\{0\}\colon T \neq \varnothing. \tag{22}$$

The model for $\text{Profit}^{AV}(N)$ indeed differs from PTP only by the inclusion of constraints (17), which ensure that all customers from S are visited (customers outside S are not visited as they are in fact not even part of the model). It is easy to see that constraint (18) is no more needed and the left-hand side of constraints (20) is never less than 2 and hence δ_T equals 0 for all nonempty $T \subset S \cup \{0\}$. The PTP-AV model could be then reformulated as

$$\text{Profit}^{AV}(S) = \max_{x_{ij}} \left(\sum_{j\in S} \text{Prize}_j - \sum_{i\in S\cup\{0\}} \sum_{j\in S\cup\{0\}} c_{ij}x_{ij} \right) \tag{23}$$

$$\text{s.t.} \quad \sum_{j\in S\cup\{0\}} x_{ij} = 1 \qquad \forall i \in S\cup\{0\}, \tag{24}$$

$$\sum_{i\in S\cup\{0\}} x_{ij} = 1 \qquad \forall j \in S\cup\{0\}, \tag{25}$$

$$\sum_{i\in T} \sum_{j\in T} x_{ij} \leq |T| - 1 \qquad \forall T \subset S\cup\{0\}\colon T \neq \varnothing, \tag{26}$$

$$x_{ij} \in \{0,1\} \qquad \forall i,j \in S\cup\{0\}. \tag{27}$$

One of two additional changes done is the replacement of constraints (16) by constraints (24) as the combination of constraints (16)-(17) is obviously equivalent to the combination of constraints (24)-(25). The second change is the change in the first term of objective function (15) which indeed holds because of constraints (17). However, this term then becomes constant over the decision variables and hence, for each S, the optimal solution is the same as for the case of TSP and the objective value is

$$\text{Profit}^{AV}(S) = \sum_{j \in S} \text{Prize}_j - \text{Cost}^{TSP}(S) \tag{28}$$

for any S.

Under what conditions does PTP generate an optimal solution with all customers in a particular coalition S being visited? Clearly, there needs to exist set $T \subseteq N \setminus S$ such that

$$\text{Profit}^{PTP} = \text{Profit}^{AV}(S \cup T). \tag{29}$$

A relationship between the objective values of PTP and PTP-AV could be expressed as

$$\text{Profit}^{PTP} = \max_{R \subseteq N} \text{Profit}^{AV}(R) \tag{30}$$

and hence

$$\text{Profit}^{AV}(S \cup T) \geq \text{Profit}^{AV}(R) \qquad \forall R \subseteq N. \tag{31}$$

Thinking about how coalition S could achieve it by setting prizes offered by customers in S (and not by the others), it needs to be noted that any customer outside S could make themselves interesting for the carrier by setting the prize exceptionally high or uninteresting or at least indifferent by setting it to zero. Therefore, as coalition S has no control over the other prizes, instead of (31), S needs to set the prizes such that, for any $T \subseteq N \setminus S$, it is profitable for the carrier to visit all customers in S, that is

$$\text{Profit}^{AV}(S \cup T) \geq \text{Profit}^{AV}(R \cup T) \qquad \forall R \subseteq S,\ \forall T \subseteq N \setminus S \tag{32}$$

or, using relation (28),

$$\sum_{j \in S \cup T} \text{Prize}_j - \text{Cost}^{TSP}(S \cup T) \geq \sum_{j \in R \cup T} \text{Prize}_j - \text{Cost}^{TSP}(R \cup T) \qquad \forall R \subseteq S,\ \forall T \subseteq N \setminus S \tag{33}$$

which can be simplified as

$$\sum_{j \in S \setminus R} \text{Prize}_j \geq \text{Cost}^{TSP}(S \cup T) - \text{Cost}^{TSP}(R \cup T) \qquad \forall R \subseteq S,\ \forall T \subseteq N \setminus S. \tag{34}$$

2.3 The profitable tour game

As discovered in the previous subsection, the carrier would visit all customers in coalition S only if the prizes were offered in a way satisfying conditions (34). It is then easy to determine the minimal total cost associated with S as

$$\text{Cost}^{PTP}(S) = \min_{\text{Prize}_j} \sum_{j \in S} \text{Prize}_j \tag{35}$$

$$\text{s.t.} \sum_{j \in S \setminus R} \text{Prize}_j \geq \text{Cost}^{TSP}(S \cup T) - \text{Cost}^{TSP}(R \cup T)$$

$$\forall R \subseteq S, \forall T \subseteq N \setminus S, \tag{36}$$

$$\text{Prize}_j \geq 0 \qquad \forall j \in S. \tag{37}$$

The pair (N, Cost^{PTP}) then defines a cooperative transferable-utility game of the customers which we name the *profitable tour game.*

Example 2. Looking back at Example 1, it is easy to compute the Cost^{PTP} values and compare them to those of Cost^{TSP}. These functions are defined for 64 different coalitions, but they differ for 10 of them only. These are reported in Table 1. Taking the first one, $S = \{1, 2, 3, 6\}$, as an example, suppose that customers 4 and 5 offer sufficiently high prizes such that the carrier would always visit them. Then, by the same logic as in Example 1, each pair of customers 1 and 2, and 3 and 6 would need to offer a total prize of at least 3 units, that adds up to at least 6 units in total. Hence, offering only 5 units does not guarantee all customers in S being visited. However, if they were to offer 6 units for instance in a way that customers 1 and 6 offer 1 unit each and 2 and 3 offer 2 units each, it would leave the carrier indifferent between visiting and not visiting all of them. Even a very small increase in the prizes would then create strong preference for visiting them. 6 units is hence indeed the minimal cost guaranteeing all customers in S being visited.

Table 1: Differences in values of cost functions Cost^{TSP} and Cost^{PTP} in Example 2

S	Cost^{TSP}	Cost^{PTP}	S	Cost^{TSP}	Cost^{PTP}
$\{1, 2, 3, 6\}$	5	6	$\{1, 2, 3, 5, 6\}$	6	7
$\{1, 2, 4, 5\}$	5	6	$\{1, 2, 4, 5, 6\}$	6	7
$\{3, 4, 5, 6\}$	5	6	$\{1, 3, 4, 5, 6\}$	6	7
$\{1, 2, 3, 4, 5\}$	7	8	$\{2, 3, 4, 5, 6\}$	7	8
$\{1, 2, 3, 4, 6\}$	7	8	$\{1, 2, 3, 4, 5, 6\}$	8	9

3 Properties of the profitable tour game

The definition of the profitable tour game provides information on costs associated with different coalitions, but prizes leading to such outcomes also deserve attention. At this point, a clear distinction between prize allocation and cost allocation needs to be made. Using game-theoretic terminology, prize allocation represents the strategies of the customers, whereas the cost allocation represents the outcome of the cooperation assigned to the customers.

When dealing with cost allocation, most studies utilize concept of the *core*. For a game (N, Cost) with $N = \{1, \ldots, n\}$, the core is defined as a set of all allocations $(x_1, \ldots, x_n)$, where x_i is the cost prescribed to be paid by customer $i \in N$, that satisfy constraints

$$\sum_{i \in N} x_i = \text{Cost}(N), \tag{38}$$

$$\sum_{i \in S} x_i \leq \text{Cost}(S) \qquad \forall S \subseteq N. \tag{39}$$

Constraint (38) guarantees that the total cost is allocated and constraints (39) ensure that no coalition can get better off by breaking the cooperation. It is important to note that for some games such allocations need not exist and then the core is empty.

3.1 Prize allocation

Many studies of traveling salesman games deal with conditions for nonemptiness of the core [13, 15]. It is then natural to study how the optimal prize allocations are affected by the emptiness of the core of the underlying traveling salesman game. In what follows, when referring to an *optimal prize allocation of the grand coalition*, we mean the prizes Prize_j that represent the optimal solution of model (35)-(37) when solved for $\text{Cost}^{PTP}(N)$.

Theorem 1. *If the core of the traveling salesman game* (N, Cost^{TSP}) *is nonempty, all allocations from the core, and no other, are optimal prize allocations of the grand coalition in the profitable tour game* (N, Cost^{PTP}).

Proof. For N, model (35)-(37) becomes

$$\text{Cost}^{PTP}(N) = \min_{\text{Prize}_j} \sum_{j \in N} \text{Prize}_j \tag{40}$$

$$\text{s.t.} \quad \sum_{j \in N \setminus R} \text{Prize}_j \geq \text{Cost}^{TSP}(N) - \text{Cost}^{TSP}(R) \qquad \forall R \subseteq N, \tag{41}$$

$$\text{Prize}_j \geq 0 \qquad \forall j \in N. \tag{42}$$

With optimal prizes, constraints (41) need to hold for all $R \subseteq N$ including $R = \varnothing$, that is

$$\sum_{j \in N} \text{Prize}_j \geq \text{Cost}^{TSP}(N). \tag{43}$$

The term on the left-hand side of this constraint is the same as in objective (40). The best solution would hence be if constraint (43) was binding, that is

$$\sum_{j \in N} \text{Prize}_j = \text{Cost}^{TSP}(N). \tag{44}$$

Assuming this to hold, constraints (41) could be rewritten as

$$\sum_{j \in R} \text{Prize}_j \leq \text{Cost}^{TSP}(R) \qquad \forall R \subseteq N. \tag{45}$$

If the core of the traveling salesman game (N, Cost^{TSP}) is nonempty, there exist prizes Prize_j that satisfy (44) and (45) and, hence, are optimal. Clearly, the set of all such prize allocations coincides with the core of (N, Cost^{TSP}). □ □

If the core contains more than just one allocation, to select a particular one, it might be useful to use allocation methods such as the nucleolus which by definition make as few constraints (45) binding as possible [11, 6]. This would contribute to lowering chances of leaving the carrier indifferent between visiting all and just some of the customers. A binding constraint (45) for a particular R leaves the carrier indifferent between visiting all customers and visiting only those in R. However, it still cannot rule out the indifference between visiting all customers and not performing a tour at all which is obvious from (44).

Example 3. Let 1, 2 and 3 be customers which desire to be visited by a carrier departing from and returning to depot 0 as illustrated in Figure 2. Each edge of the graph is accompanied by a number standing for the respective traveling cost. The traveling salesman problem can be solved to obtain costs for different coalitions. This results in $\text{Cost}^{TSP}(\varnothing) = 0$, $\text{Cost}^{TSP}(\{1\}) = \text{Cost}^{TSP}(\{2\}) = \text{Cost}^{TSP}(\{3\}) = 20$, $\text{Cost}^{TSP}(\{1,2\}) = \text{Cost}^{TSP}(\{1,3\}) = \text{Cost}^{TSP}(\{2,3\}) = 37$, $\text{Cost}^{TSP}(\{1,2,3\}) = 54$.

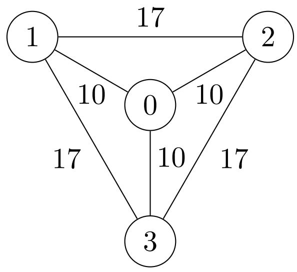

Figure 2: Problem in Example 3

By Theorem 1, optimal prize allocation $(\text{Prize}_1, \text{Prize}_2, \text{Prize}_3)$ belongs to the core of game $(\{1,2,3\}, \text{Cost}^{TSP})$, that is

$$\text{Prize}_1 + \text{Prize}_2 + \text{Prize}_3 = 54, \tag{46}$$

$$\text{Prize}_1 + \text{Prize}_2 \leq 37, \tag{47}$$

$$\text{Prize}_1 + \text{Prize}_3 \leq 37, \tag{48}$$

$$\text{Prize}_2 + \text{Prize}_3 \leq 37, \tag{49}$$

$$\text{Prize}_1 \leq 20, \tag{50}$$

$$\text{Prize}_2 \leq 20, \tag{51}$$

$$\text{Prize}_3 \leq 20. \tag{52}$$

This is satisfied for example by $\text{Prize}_1 = 20$, $\text{Prize}_2 = 17$, $\text{Prize}_3 = 17$. However, one can observe that it makes the carrier indifferent between visiting all customers, visiting customers 1 and 2, visiting customers 1 and 3, visiting only customer 1, and visiting no customers. However, allocating the prizes as prescribed by the nucleolus, that is $\text{Prize}_1 = 18$, $\text{Prize}_2 = 18$, $\text{Prize}_3 = 18$, would leave the carrier indifferent only between visiting all customers and visiting no customers. Even a marginal increase in any of these prizes would then incentivize the carrier to perform the tour visiting all customers.*

Theorem 1 implies $\mathrm{Cost}^{PTP}(N) = \mathrm{Cost}^{TSP}(N)$. This was not the case for Examples 1 and 2 which necessarily means that the core of the underlying traveling salesman game is empty for this example. This is indeed shown by Tamir [15].

Examples 1 and 2 might look strange as for many combinations of the traveling costs the triangle inequality becomes equality and the unit traveling costs are not represented in Figure 1 with edges of the same length. Different explanations of this might take place such as an underlying road network does not allow shorter ways between some customers and a short distance traveled on a bad quality road is as expensive as a long distance traveled on a good quality road. However, there are also examples of cases with Euclidean distance between all places which result in games with empty cores [2].

Theorem 2. *If the core of the traveling salesman game* (N, Cost^{TSP}) *is empty, then prizes* Prize_j *stand for an optimal prize allocation of the grand coalition in the profitable tour game* (N, Cost^{PTP}) *if and only if they represent an optimal solution of problem*

$$\min_{\mathrm{Prize}_j,\varepsilon} \ \varepsilon \tag{53}$$

$$\text{s.t.} \quad \sum_{j\in N} \mathrm{Prize}_j = \mathrm{Cost}^{TSP}(N) + \varepsilon, \tag{54}$$

$$\sum_{j\in S} \mathrm{Prize}_j \leq \mathrm{Cost}^{TSP}(S) + \varepsilon \qquad \forall S \subset N, \tag{55}$$

$$\mathrm{Prize}_j \geq 0 \qquad \forall j \in N, \tag{56}$$

$$\varepsilon \geq 0. \tag{57}$$

Proof. Following the same path as in the proof of Theorem 1, it is clear that assumption (44) would not be correct in this case as there exist no prizes Prize_j that satisfy (44) and (45) when the core of the traveling salesman game (N, Cost^{TSP}) is empty.

Assuming instead

$$\sum_{j\in N} \mathrm{Prize}_j = \mathrm{Cost}^{TSP}(N) + \varepsilon. \tag{58}$$

for an arbitrary value of ε, constraints (41) could be reformulated as

$$\sum_{j\in R} \mathrm{Prize}_j \leq \mathrm{Cost}^{TSP}(R) + \varepsilon \qquad \forall R \subseteq N. \tag{59}$$

*A question could be raised of which ones of the customers should increase the prize. This might open a long discussion since arguably every individual customer wants to minimize its own prize. However, in the cooperative game-theoretical framework we adopt, the increment could be already reflected in the cost function value $\mathrm{Cost}^{TSP}(\{1,2,3\})$. Thus, the increment gets allocated in a manner that is fair according to the chosen allocation method (the nucleolus in this example).

Lastly, because the term on the left-hand side of constraint (58) is the same as in objective function (40) and $\text{Cost}^{TSP}(N)$ is constant over the decision variables, then, in terms of the optimal solution for prizes Prize_j, objective function (40) is equivalent to

$$\min_{\text{Prize}_j, \varepsilon} \ \varepsilon. \tag{60}$$

.

Model (40)-(42) can then be reformulated as (53)-(57) while preserving the same optimal solution for prizes Prize_j. □ □

It is important to note that model (53)-(57) is always feasible. For example, prizes prescribed as $\text{Prize}_j = \frac{\text{Cost}^{TSP}(N)+\varepsilon}{|N|}$ for each $j \in N$ clearly satisfy constraint (54) and, since $\text{Cost}^{TSP}(S) \geq 0$ for any $S \subseteq N$, then $\varepsilon \geq |N| \, \text{Cost}^{TSP}(N)$ guarantees satisfaction of constraints (55)-(57) as well.

Whereas there might exist multiple optimal solutions for prizes Prize_j, optimal ε is obviously unique. Then, game $(N, \widehat{\text{Cost}}^{TSP})$ can be defined, where $\widehat{\text{Cost}}^{TSP}(S) = \text{Cost}^{TSP}(S) + \varepsilon$ for each $S \subseteq N$. Straightforwardly, all allocations from the core of $(N, \widehat{\text{Cost}}^{TSP})$, and no other, are optimal prize allocations of the grand coalition in profitable tour game (N, Cost^{PTP}). This allows for usage of allocation methods such as the nucleolus for problems with empty cores of the associated traveling salesman games with the same implications as in the case of nonempty cores.

Example 4. Solving model (53)-(57) for Examples 1 and 2 results in $\varepsilon = 1$. Thanks to Example 2, we actually already knew the value of ε because

$$\varepsilon = \text{Cost}^{PTP}(N) - \text{Cost}^{TSP}(N) = 1. \tag{61}$$

Defining $\widehat{\text{Cost}}^{TSP}$ such that $\widehat{\text{Cost}}^{TSP}(S) = \text{Cost}^{TSP}(S) + 1$ for each $S \subseteq N$ and analyzing the core of the respective game introduces a system of one equality and 62 inequalities describing the set of all optimal prize allocations. To choose just one of them, the nucleolus prescribes the prize allocation $\text{Prize}_1 = 1$, $\text{Prize}_2 = 2$, $\text{Prize}_3 = 2$, $\text{Prize}_4 = 2$, $\text{Prize}_5 = 1$, and $\text{Prize}_6 = 1$.

Note that Theorem 2 could be generalized to all profitable tour games regardless of the core emptiness of the respective traveling salesman game. In fact, if the core of the traveling salesman game (N, Cost^{TSP}) is nonempty, the optimal value of ε in problem (53)-(57) equals 0 and the optimal prizes Prize_j must satisfy (44) and (45) which define the core of (N, Cost^{TSP}) as in Theorem 1.

An interesting corollary of Theorems 1 and 2 appears when utilizing a concept of the *dual game* [12, 16]. For a game (N, Cost), the dual game is defined as a game $(N, \text{Cost*})$ where

$$\text{Cost*}(S) = \text{Cost}(N) - \text{Cost}(N \setminus S) \qquad \forall S \subseteq N. \tag{62}$$

The corollary can be stated without a proof as it follows directly from model (40)-(42).

Corollary 3. *Prizes* Prize_j *stand for an optimal prize allocation of the grand coalition in the profitable tour game* (N, Cost^{PTP}) *if and only if they represent an optimal solution of problem*

$$\min_{\text{Prize}_j} \sum_{j \in N} \text{Prize}_j \tag{63}$$

$$\text{s.t.} \quad \sum_{j \in R} \text{Prize}_j \geq \text{Cost}^{TSP}*(R) \qquad \forall R \subseteq N, \tag{64}$$

$$\text{Prize}_j \geq 0 \qquad \forall j \in N. \tag{65}$$

where $\text{Cost}^{TSP}*$ *represents a cost function of the dual game to the traveling salesman game* (N, Cost^{TSP})*.*

4 Conclusion

We have studied the profitable tour problem where a profit-maximizing carrier decides whether to visit a particular customer with respect to the prize the customer offers for being visited and traveling cost associated with the visit, all in the context of other customers. Our focus has been on the prizes that customers need to offer to ensure being visited by the carrier. This can be formulated as a cooperative game where customers may form coalitions and make decisions on the prize values cooperatively. We have defined the profitable tour game describing the situation in which customers need to compete or cooperate and make themselves worth being visited. This problem relates to logistics applications such as the evaluation of new customers in traveling salesman problems or customer selection in less-than-truckload transportation.

We have found several properties of the optimal prizes to be offered if the coalition of all customers, the grand coalition, is formed. These properties describe a relationship between the prizes and the underlying traveling salesman game to provide another connection with an extensive literature on core allocations in traveling salesman games. Our most important result states that the set of optimal prizes to be offered coincides with the core of the underlying traveling salesman game if this core is nonempty. Other results include the optimal prizes description for the empty-core case and relation of the prizes to the dual game of the traveling salesman game.

Overall, we have analyzed the total cost associated with each coalition of customers and, in form of the prize allocation, the strategies to achieve it. Our analysis opens further research opportunities for studying cost allocation to divide the costs paid out in form of prizes among the cooperating customers. Analyzing the core of the profitable tour game and the performance of specific allocation methods are interesting avenues for future research.

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